CAUCHY INTEGRAL

Theorem (Analyticity of Cauchy Integral)**.** *Let γ be a piece-wise smooth curve and φ be a piecewise continuous bounded function on γ.*

For $z \notin \gamma$, define

$$F(z) := \oint_\gamma \frac{\varphi(\xi)}{\xi - z}\, d\xi. \tag{1}$$

If $z_0 \notin \gamma$, then for all $z \notin \gamma$, $F(z)$ can be represented as:

$$F(z) = \sum_{k=0}^{n-1} a_k (z - z_0)^k + F_n(z)(z - z_0)^n, \tag{2}$$

$$\text{where } F_n(z) = \oint_\gamma \frac{\varphi(\xi)}{(\xi - z)^n(\xi - z)}\, d\xi \text{ and } a_k = \oint_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^{k+1}}\, d\xi.$$

Moreover, for all $z \in \mathbb{C}$ with $|z - z_0| < \operatorname{dist}(z_0, \gamma)$, we can represent $F(z)$ as:

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{3}$$

Remark. *The identity (3) implies that the function $F(z_0 + w)$ is the sum of the power series $\sum_{n=0}^{\infty} a_n w^n$ for all w with $|w| < \operatorname{dist}(z_0, \gamma)$. By Taylor Theorem, F is infinitely differentiable at z_0 and $a_n = \frac{F^{(n)}(z_0)}{n!}$.*

As a bonus, we get an integral formula for the derivative

$$F^{(n)}(z_0) = n! \oint_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$$

Moreover, the equation (2) becomes a representation of F as a sum of its Taylor polynomial and a reminder in **Cauchy form***.*

Proof of the Theorem. First, fix $z_0 \notin \gamma$, $z \notin \gamma$ and $\xi \in \gamma$. Note that $z \neq \xi$, so $\frac{z - z_0}{\xi - z_0} \neq 1$. By the formula for the finite geometric series, we have for any $n \in \mathbb{N}$

$$\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{k=0}^{n-1} \left(\frac{z - z_0}{\xi - z_0} \right)^k + \frac{\left(\frac{z - z_0}{\xi - z_0} \right)^n}{1 - \frac{z - z_0}{\xi - z_0}}$$

which is equivalent to

$$\frac{\xi - z_0}{\xi - z} = \sum_{k=0}^{n-1} \left(\frac{z - z_0}{\xi - z_0} \right)^k + \frac{(z - z_0)^n}{(\xi - z_0)^{n-1}(\xi - z)},$$

and, after dividing by $\xi - z_0$, we get

$$\boxed{\frac{1}{\xi - z} = \sum_{k=0}^{n-1} \frac{(z - z_0)^k}{(\xi - z_0)^{k+1}} + \frac{(z - z_0)^n}{(\xi - z_0)^n(\xi - z)}} \tag{4}$$

We can multiply the identity (4) by $\varphi(\xi)$ and integrate over γ to get

$$(5) \quad F(z) = \oint_\gamma \frac{\varphi(\xi)}{\xi - z}\, d\xi = \sum_{k=0}^{n-1} \oint_\gamma (z - z_0)^k \frac{\varphi(\xi)}{(\xi - z_0)^{k+1}}\, d\xi + \oint_\gamma (z - z_0)^n \frac{\varphi(\xi)}{(\xi - z_0)^n(\xi - z)}\, d\xi$$

This is exactly the identity (2).

To obtain (3), we just need to show that for any $z \in \mathbb{C}$ with $|z - z_0| < \mathrm{dist}(z_0, \gamma)$ we have

$$\lim_{n\to\infty} (z - z_0)^n F_n(z) = \lim_{n\to\infty} \oint_\gamma (z - z_0)^n \frac{\varphi(\xi)}{(\xi - z_0)^n(\xi - z)}\, d\xi = 0.$$

Let $q := \frac{|z-z_0|}{R}$. Observe that $q < 1$. Note that

(1) $\frac{|z-z_0|}{|\xi-z_0|} \leq q$,

(2) $|\xi - z_0| \geq R$,

(3) $|\xi - z| \geq |\xi - z_0| - |z - z_0| \geq R - |z - z_0|$

(4) Since φ is bounded on γ, for some M and all $\xi \in \gamma$ we have$|\varphi(\xi)| \leq M$.

Applying all these estimates to the integral formula for $F_n(z)$ we get

$$|z - z_0|^n |F_n(z)| \leq q^n \frac{1}{R} \frac{M}{R - |z - z_0|}\, \mathrm{length}(\gamma) \to 0.$$

$\square$